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Pseudo supersymmetric partners for the generalized Swanson model

A Sinha and P Roy

Physics and Applied Mathematics Unit Indian Statistical Institute Kolkata-700 108, India

E-mail: anjana_t@isical.ac.in and pinaki@isical.ac.in

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Abstract

New non Hermitian Hamiltonians are generated, as isospectral partners of the generalized Swanson model, viz., $H_- = \mathcal{A}^\dagger\mathcal{A} + \alpha\mathcal{A}^2 + \beta\mathcal{A}^{\dagger 2}$, where α, β are real constants, with $\alpha \neq \beta$, and $\mathcal{A}^\dagger$ and $\mathcal{A}$ are generalized creation and annihilation operators. It is shown that the initial Hamiltonian H_-, and its partner H_+, are related by pseudo supersymmetry, and they share all the eigen energies except for the ground state. This pseudo supersymmetric extension enlarges the class of non Hermitian Hamiltonians $H_\pm$, related to their respective Hermitian counterparts $h_\pm$, through the same similarity transformation operator $\rho : H_\pm = \rho^{-1}h_\pm\rho$. The formalism is applied to the entire class of shape-invariant models.

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1. Introduction

Ever since interest in non Hermitian Hamiltonians (with real energies) was revived about a decade ago by Bender and Boettcher [1], quantum systems described by such non Hermitian Hamiltonians have been studied widely [2]. To extend the class of such systems, new exactly solvable (or quasi-exactly solvable) non Hermitian Hamiltonians with real, discrete energies, have been generated using different approaches—e.g., supersymmetry [3], the related intertwining operator method [4], or the Darboux algorithm [5]. In a recent work [6], we had found a class of new non Hermitian models by generalizing the Swanson Hamiltonian $H = a^\dagger a + \alpha a^2 + \beta a^{\dagger 2}$ where α, β are real constants, with $\alpha \neq \beta$. This model was initially proposed by Swanson [7], and later on studied by various authors [8]. For the sake of generalization, we had used generalized creation and annihilation operators, $\mathcal{A}^\dagger$ and $\mathcal{A}$, in place of Harmonic oscillator creation and annihilation operators $a^\dagger, a$, so that $H_- = \mathcal{A}^\dagger\mathcal{A} + \alpha\mathcal{A}^2 + \beta\mathcal{A}^{\dagger 2}$. The energies of this class of non Hermitian Hamiltonians were found to be real when the parameters satisfy the relations $(\alpha + \beta) < 1$ and $4\alpha\beta < 1$ [6, 7]. In the present work, we shall generate new non Hermitian Hamiltonians H_+, as isospectral partners of the generalized Swanson Hamiltonian H_-. It may be recalled that non Hermiticity may be introduced through a *scalar* term or by a *vector* term. In the *scalar* case non Hermiticity may be introduced by replacing x with $(x \pm i\epsilon)$ or by taking one of the parameters complex in the expression for the potential $V(x)$. Though this looks apparently simple, nevertheless, a similarity transformation of the non Hermitian Hamiltonian ($H_\pm$ with solutions $\psi^{(\pm)}$) maps it to a complicated non-local Hermitian Hamiltonian ($h_\pm$ with solutions $\phi^{(\pm)}$). Consequently, the resulting Hermitian Hamiltonian is not exactly solvable and perturbative techniques may have to be applied for solving the same. Additionally, it may not be possible to determine the exact form of the metric operator η explicitly, with respect to which the inner product $\langle\psi_m|\eta|\psi_n\rangle$ is positive definite. On the contrary, if non Hermiticity is introduced through an imaginary *vector* potential, the similarity transformation yields a Hermitian Schrödinger Hamiltonian (consisting of the standard kinetic term plus a local Hermitian potential term), with possibilities of exact (or quasi-exact) solvability. The metric operator also can be obtained in closed form. In view of the fact that a gauge-like transformation transforms the non Hermitian Hamiltonian H_- to a Hermitian one h_- with Schrödinger form, the imaginary vector potential may be regarded as a trivial way of introducing non Hermiticity. Nevertheless, since the resulting Hamiltonian H_- comes out to be real yet non Hermitian, with real spectrum and possibilities of exact (or quasi-exact) solutions, our interest here is to look for isospectral partners of such a potential.

A couple of recent works deserve special mention here [9, 10]. In the first of these [9], a method was proposed to generate a family of non Hermitian Hamiltonians equivalent to the Swanson Hamiltonian [7], by writing H as a linear combination of $su(1,1)$ generators K_0, $K_\pm$; i.e. $H = 2K_0 + 2\alpha K_- + 2\beta K_+$. In the second work [10], a quasi-Hermitian supersymmetric extension was proposed for a Harmonic Oscillator Hamiltonian, augmented by a non Hermitian $\mathcal{PT}$ symmetric part. To construct new non Hermitian Hamiltonians related by similarity transformation to Hermitian ones [11, 12], the $su(1,1)$ Lie algebra needs to be enlarged to a $su(1,1/1) \sim \mathrm{osp}(2/2,\mathbf{R})$ Lie super algebra. Incidentally, both in [10] and our present formulation, non Hermiticity is introduced not by considering a complex-valued potential, but through a momentum-dependent interaction term. However, our approach is different from either of those employed in [9, 10]. Instead of the Swanson model considered in [9, 10], we deal with its generalized version [6]. So while the non Hermiticity is introduced through a momentum-dependent *linear* interaction term in [10], viz., $i(\alpha-\beta)(xp+px)$; in our case it is not necessarily linear, to be precise it is of the form $i(\alpha-\beta)\{W(x)p + pW(x)\}$, thus depending on the particular model considered. Secondly, in our case the Hamiltonian H_- is written in terms of generalized creation and annihilation operators, which are not necessarily Lie algebra generators.

It is worth recalling here that the choice of the metric operator η is not unique. In fact many possible such operators exist, obeying the condition $\eta H_- = h_-^\dagger\eta$. So there are many ways of finding a Hermitian Hamiltonian h_-, through a similarity transformation $h_- = \rho H_- \rho^{-1}$ and thus obtain the metric operator η [13], from the relation $\eta = \rho^2$ [11]. Each h_- is associated with a different metric, thus invoking a different Hilbert space for each Hermitian map. Our approach gives a simple, straightforward method to determine one such similarity transformation ρ mapping the non Hermitian system H_- to its Hermitian equivalent h_-. In this respect our approach is different from [10] where the $su(1,1)$ generators K_0, $K_\pm$ are used in the construction of ρ. Furthermore, the normalization requirement of the wave functions for pseudo Hermitian systems viz., $\langle\psi|\eta|\psi\rangle$ ensures the wave functions to be naturally normalized in our formalism, as we shall see later.

Once a non Hermitian partner Hamiltonian H_+ of the generalized Swanson Hamiltonian H_- is obtained, it is natural to look for some underlying symmetry between $H_\pm$. Since our starting Hamiltonian is non Hermitian, the partners cannot be expected to be inter-related through spersymmetry. On the contrary, it is anticipated that they will be related by pseudo supersymmetry [14]. It will also be shown that the pair of non Hermitian Hamiltonians $H_\pm$ are related to a pair of Hermitian ones $h_\pm$ through the same similarity transformation ρ. Finally, we shall apply our formalism to the entire class of shape-invariant potentials, where the parameters of the partner potential are related to those of the initial one through translation [15]. It is worth mentioning here that we have been able to give a general expression for finding the partner Hamiltonian H_+ in terms of the parameters of H_-, for all the shape-invariant models related through translation of parameters.

The plan of the work is as follows. In section 2, new non Hermitian Hamiltonians H_+ are generated, which are isospectral to the initial non Hermitian Hamiltonian H_-, except for the ground state. It is observed further that both the initial non Hermitian Hamiltonian H_- and its partner H_+ so generated, are pseudo Hermitian with respect to the same linear, invertible operator η. The underlying symmetry between the partners $H_\pm$ is studied in section 3. The formalism developed here is actually applied to all the known classes of shape-invariant models mentioned above, in section 4. Finally, Section 5 is kept for conclusions and discussions.

2. Theory

For a better understanding of the topic and to make the paper self-contained, we repeat certain equations from [6] in the initial part of this section. To start with we consider the generalized Swanson model

$$H_- = \mathcal{A}^\dagger\mathcal{A} + \alpha\mathcal{A}^2 + \beta\mathcal{A}^{\dagger 2}, \qquad \alpha \neq \beta \tag{1}$$

where α and β are real, dimensionless constants, with $\alpha \neq \beta$ for H_- to be non Hermitian, and $\mathcal{A}^\dagger$ and $\mathcal{A}$ are generalized creation and annihilation operators, given by

$$\mathcal{A} = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \qquad \mathcal{A}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{2}$$

Investigations in this field has revealed that for such non Hermitian Hamiltonians to describe physical systems, they should be necessarily η-pseudo Hermitian [13],

$$H_-^\dagger = \eta H_- \eta^{-1}, \qquad \text{i.e.} \quad H_-^\dagger\eta = \eta H_- \tag{3}$$

where η is a linear, invertible, Hermitian operator. This requirement, along with the criterion for the wave functions to be well behaved in the entire range, the parameters must obey certain conditions [6, 7], viz.,

$$\alpha + \beta < 1, \qquad 4\alpha\beta < 1 \tag{4}$$

With the explicit form of (2) and some straightforward algebra, the eigenvalue equation

$$H_-\psi^{(-)}(x) = E\psi^{(-)}(x) \tag{5}$$

can be cast in the form [6]

$$\begin{aligned} H_-\psi^{(-)} &= \left\{-(1-\alpha-\beta)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\alpha-\beta}{1-\alpha-\beta}W\right)^2 + \frac{1-4\alpha\beta}{(1-\alpha-\beta)}W^2 - W'\right\}\psi^{(-)} \\ &= E\psi^{(-)} \end{aligned} \tag{6}$$

To reduce equation (6) to the well known Schrödinger form

$$h_-\phi^{(-)}(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x)\right)\phi^{(-)}(x) = \varepsilon\phi^{(-)}(x) \tag{7}$$

one has to apply a transformation of the form [16]

$$\psi^{(-)}(x) = \rho^{-1}\phi^{(-)}(x) \tag{8}$$

where

$$\rho = \mathrm{e}^{-\mu\int W(x)\mathrm{d}x}, \qquad \text{with} \qquad \mu = \frac{\alpha-\beta}{1-\alpha-\beta}, \quad \alpha+\beta \neq 1 \tag{9}$$

so that comparison between (6) and (7) gives

$$\begin{aligned} V_-(x) &= \left(\frac{\sqrt{1-4\alpha\beta}}{1-\alpha-\beta}W(x)\right)^2 - \frac{1}{(1-\alpha-\beta)}W'(x) \\ \varepsilon &= \frac{E}{1-\alpha-\beta} \end{aligned} \tag{10}$$

Thus, a quantum system described by a pseudo Hermitian Hamiltonian H_-, is mapped to an equivalent system described by its corresponding Hermitian counterpart h_-, with the help of a similarity transformation ρ [6, 11, 12],

$$h_- = \rho H_- \rho^{-1} \tag{11}$$

We, now, take refuge in the formalism of supersymmetric quantum mechanics (SUSYQM) [3], or the equivalent intertwining operator method [4], to find an isospectral partner of h_-. As is well known, h_- can always be written in a factorizable form as a product of a pair of linear differential operators $\tilde{A}$, $\tilde{A}^\dagger$, as

$$h_- = \tilde{A}^\dagger\tilde{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + w^2 - w' \tag{12}$$

apart from some factorization energy ϵ, where $\tilde{A}$, $\tilde{A}^\dagger$ and $w(x)$ are given by

$$\tilde{A} = \frac{\mathrm{d}}{\mathrm{d}x} + w(x), \qquad \tilde{A}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + w(x), \qquad w(x) = -\frac{\mathrm{d}\ln\phi_0^-(x)}{\mathrm{d}x} \tag{13}$$

ϕ_0^- being the ground state eigenfunction of $\tilde{A}^\dagger\tilde{A}$ with energy ε_0. Thus $V_-(x)$ in (10) can be identified with $(w^2 - w')$

$$V_-(x) = w^2(x) - w'(x) \tag{14}$$

With the help of (10), the original eigenvalue equation (6) may be written in a more compact form as

$$H_-\psi^{(-)}(x) = (1-\alpha-\beta)\left\{-\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\alpha-\beta}{1-\alpha-\beta}W(x)\right)^2 + V_-(x)\right\}\psi^{(-)}(x) = E\psi^{(-)}(x) \tag{15}$$

By the principles of SUSYQM, the hamiltonian h_- is isospectral to its partner Hamiltonian h_+ given by

$$h_+ = \tilde{A}\tilde{A}^\dagger = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + w^2 + w' \tag{16}$$

i.e.,

$$h_+\phi^{(+)}(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x)\right)\phi^{(+)}(x) = \varepsilon\phi^{(+)}(x) \tag{17}$$

where

$$V_+(x) = w^2(x) + w'(x) \tag{18}$$

Let us now apply the inverse transformation of that given in (8) to (17) above, i.e.,

$$\phi^{(+)}(x) = \rho\psi^{(+)}(x) = \mathrm{e}^{-\mu\int W(x)\mathrm{d}x}\psi^{(+)}(x) \tag{19}$$

After some straightforward algebra, equation (17) can be written as

$$H_+\psi^{(+)} = (1-\alpha-\beta)\left\{-\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\alpha-\beta}{1-\alpha-\beta}W(x)\right)^2 + V_+(x)\right\}\psi^{(+)} = E\psi^{(+)} \tag{20}$$

Thus, $H_\pm$ are of the same form, except for the explicit form of $V_\pm(x)$. Evidently, both the initial Hamiltonian H_- as well as its partner H_+ are non Hermitian. Since $h_\pm$ share identical energies, except for the ground state, so should $H_\pm$, with the exception of the ground state. Thus, applying the principles of SUSYQM, we obtain a non Hermitian partner Hamiltonian H_+ of the initial one H_-, sharing identical energies except for the ground state.

2.1. Pseudo Hermiticity of H_+

If one considers the inverse transformation (19), then it is easy to check that both the Hermitian Hamiltonian $h_\pm$ and their non Hermitian counterparts $H_\pm$ are related by the same similarity transformation as in (11), i.e.,

$$H_\pm = \rho^{-1}h_\pm\rho \tag{21}$$

Additionally, simple algebra shows that both the non Hermitian Hamiltonian $H_\pm$ are pseudo Hermitian with respect to the same pseudo Hermiticity operator η

$$H_\pm^\dagger = \eta H_\pm\eta^{-1} \qquad \text{i.e.} \quad H_\pm^\dagger\eta = \eta H_\pm \tag{22}$$

where ρ and η are inter-related through $\rho = \sqrt{\eta}$ [6, 11].

It is interesting to study the behaviour of the wave functions $\psi^{(\pm)}(x)$. Since $H_\pm$ are η-pseudo Hermitian, the wave functions should be normalized as $\langle\psi^{(\pm)}|\eta|\psi^{(\pm)}\rangle$ [13]. With $\eta = \rho^2$ and $\psi^{(\pm)}(x) = \rho^{-1}\phi^{(\pm)}(x)$, the above normalization condition reduces to the conventional normalization of Hermitian quantum systems, viz., $\langle\phi^{(\pm)}|\phi^{(\pm)}\rangle$, easily available in standard text books of quantum mechanics for the shape-invariant potentials considered here [3].

3. Underlying symmetry between the partners $H_\pm$

To explore the underlying symmetry between the isospectral partners $H_\pm$, we start with their Hermitian counterparts $h_\pm$. Now, $h_\pm$ form a pair of supersymmetric partners, with super Hamiltonian

$$h = \begin{pmatrix} h_- & 0 \\ 0 & h_+ \end{pmatrix} \tag{23}$$

and generated by supercharges

$$q = \begin{pmatrix} 0 & \tilde{A}^\dagger \\ 0 & 0 \end{pmatrix}, \qquad q^\dagger = \begin{pmatrix} 0 & 0 \\ \tilde{A} & 0 \end{pmatrix} \tag{24}$$

so that

$$h = \{q^\dagger, q\} \tag{25}$$

To establish the symmetry relation between $H_\pm$, we return to the similarity transformation between the original non Hermitian Hamiltonian H_- and its Hermitian mapping h_-, viz.,

$$H_- = \rho^{-1}h_-\rho$$

If one defines two operators $D_\pm$ [1] as

$$D_+ = (\sqrt{1-\alpha-\beta})\rho^{-1}\tilde{A}^\dagger\rho \qquad D_- = (\sqrt{1-\alpha-\beta})\rho^{-1}\tilde{A}\rho \tag{26}$$

then the isospectral Hamiltonians, $H_\pm$, can be written in terms of these operators as

$$H_- = D_+D_-, \qquad H_+ = D_-D_+ \tag{27}$$

so that $D_\pm$ play the role of intertwining operators for $H_\pm$

$$D_-H_- = H_+D_-, \qquad H_-D_+ = D_+H_+ \tag{28}$$

With the help of (2) and (26), $D_\pm$ can be written in the explicit form

$$\begin{aligned} D_+ &= (\sqrt{1-\alpha-\beta})\left\{-\frac{\mathrm{d}}{\mathrm{d}x} + \mu W(x) + w(x)\right\} \\ D_- &= (\sqrt{1-\alpha-\beta})\left\{\frac{\mathrm{d}}{\mathrm{d}x} - \mu W(x) + w(x)\right\} \end{aligned} \tag{29}$$

It is worth noting here that the functions $W(x)$ and $w(x)$ appearing in the explicit form of $D_\pm$ are not independent. Instead, they are related to each other by equations (10) and (14), i.e.

$$w^2(x) - w'(x) = \left(\frac{\sqrt{1-4\alpha\beta}}{1-\alpha-\beta}W(x)\right)^2 - \frac{1}{(1-\alpha-\beta)}W'(x) \tag{30}$$

Since the isospectral partner Hamiltonians $H_\pm$ are pseudo Hermitian, we expect them to be embedded in the framework of pseudo supersymmetry [14]. Straightforward algebra shows that the operators $D_\pm$ are pseudo-adjoint of one another

$$(D_+)^\sharp = \eta^{-1}(D_+)^\dagger\eta = \eta^{-1}\left(\rho\tilde{A}\rho^{-1}\right)\eta = \rho^{-1}\tilde{A}\rho = D_- \tag{31}$$

If we define two operators Q and $Q^\sharp$ as

$$Q = \begin{pmatrix} 0 & D_+ \\ 0 & 0 \end{pmatrix}, \qquad Q^\sharp = \eta^{-1}Q^\dagger\eta = \begin{pmatrix} 0 & 0 \\ D_- & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (D_+)^\sharp & 0 \end{pmatrix} \tag{32}$$

and construct a new Hamiltonian $\mathcal{H}$ from the partners $H_\pm$ as

$$\mathcal{H} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \tag{33}$$

then it is easy to observe that

$$\mathcal{H} = \{Q^\sharp, Q\} \tag{34}$$

Additionally,

$$\{Q, Q\} = \{Q^\sharp, Q^\sharp\} = 0 \tag{35}$$

Thus we obtain the standard pseudo super algebra of non Hermitian supersymmetry [14], with the operators Q and $Q^\sharp$ playing the role of pseudo super charges, the anticommutator of which gives the pseudo super Hamiltonian $\mathcal{H}$. Interestingly, though it may not be possible (in general) to express the new Hamiltonian H_+ in terms of the generalized annihilation and creation operators $\mathcal{A}$ and $\mathcal{A}^\dagger$, nevertheless, the isospectral partners $H_\pm$ can be shown to be related by pseudo supersymmetry. Furthermore, it is also observed that the super charges $q, q^\dagger$ of conventional supersymmetry are related to the pseudo supercharges $Q, Q^\sharp$ of pseudo supersymmetry through

$$Q = \rho^{-1}q\rho \tag{36}$$

[1] In case the operators $D_\pm$ are defined by taking the negative square root of $(1-\alpha-\beta)$, then the new operators $D_\pm^{\mathrm{new}}$ so formed are related to $D_\pm$ through a constant phase, viz., $D_\pm^{\mathrm{new}} = \mathrm{e}^{\pm\mathrm{i}\pi}D_\pm$. This introduces no change in the expression for $H_\pm$.

This follows from the similarity mapping between the non Hermitian Hamiltonians $H_\pm$ and their respective Hermitian counterparts $h_\pm$. We shall devote the next section to construct some non Hermitian Hamiltonians as isospectral partners of the generalized Swanson models based on those shape-invariant potentials where the parameters are related to each other by translation ($a_2 = a_1 + \lambda$) [3, 15].

4. Models based on shape-invariant potentials

For our formalism to be applicable to specific models, one needs to solve the highly non-trivial Ricatti equation (30). This demands certain restrictions on the forms of $W(x)$ and $w(x)$. If one wants to map a certain type of potential (say Harmonic oscillator) to a different type (say e.g., Pöschl–Teller or Rosen–Morse), while going from the non Hermitian to the Hermitian picture, the corresponding Ricatti equation cannot be solved analytically (or, at least in an obvious way). For the shape invariant class, the function $w(x)$ consists of two parts, denoted by $f(x)$ and $g(x)$, i.e.

$$w(x) = \lambda_1 f(x) + \delta_1 g(x), \qquad \text{with} \quad \lambda_1, \delta_1 \text{ constants} \tag{37}$$

For reasons given in the beginning of this section, the function $W(x)$ used in the construction of generalized annihilation and creation operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ in (2) is assumed to be of the same form as $w(x)$:

$$W(x) = \lambda_2 f(x) + \delta_2 g(x), \qquad \text{with} \quad \lambda_2, \delta_2 \text{ constants} \tag{38}$$

Our aim is to write $V_-(x)$ in terms of $w^2(x) - w'(x)$. It is already shown that $W(x)$ and $w(x)$ are inter-related through (30). Substituting (37) and (38) in (30), the expression takes the explicit form

$$\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}\left\{\lambda_2^2 f^2 + \delta_2^2 g^2 + 2\lambda_2\delta_2 fg\right\} - \frac{1}{1-\alpha-\beta}(\lambda_2 f' + \delta_2 g')$$
$$= \lambda_1^2 f^2 + \delta_1^2 g^2 + 2\lambda_1\delta_1 fg - \lambda_1 f' - \delta_1 g' \tag{39}$$

This general expression relates the unknown parameters λ_1, δ_1 in terms of the known ones λ_2, δ_2, for all shape invariant potentials where the parameters of the original potential and its partner are related to each other by translation. This enables one to write the partner potential $V_+(x)$, and hence the partner Hamiltonian H_+, in terms of the parameters of the starting Hamiltonian H_-. Now these shape-invariant models can be further classified under different categories, depending on the particular forms of $f(x)$ and $g(x)$. We shall explore these in further detail in the next few subsections.

4.1. *Case 1:* $g(x) = \text{constant}$, $f^2(x) = c_1 f'(x) + c_2$, *with* c_1, c_2 *constants*

In this subsection, we shall study the models based on the following potentials:

(1) *Rosen–Morse I (trigonometric) potential*

$$V(x) = a(a-1)\csc^2 x + 2b\cot x - a^2 + \frac{b^2}{a^2}, \qquad 0 \leqslant x \leqslant \pi \tag{40}$$

$$\text{with} \quad W(x) = -a_2\cot x - \frac{b_2}{a_2}, \qquad a_2 > 0, \quad b_2 > 0 \tag{41}$$

(2) *Rosen–Morse II (hyperbolic) potential*

$$V(x) = -a(a+1)\mathrm{sech}^2 x + 2b\tanh x + a^2 + \frac{b^2}{a^2}, \qquad b < a^2, \quad -\infty \leqslant x \leqslant \infty \tag{42}$$

$$\text{with} \quad W(x) = a_2 \tanh x + \frac{b_2}{a_2}, \qquad a_2 > 0, \quad b_2 > 0 \tag{43}$$

(3) *Eckart potential*

$$V(x) = a(a-1)\,\mathrm{cosech}^2 x - 2b\coth x + a^2 + \frac{b^2}{a^2}, \qquad b > a^2, \quad 0 \leqslant x \leqslant \infty \tag{44}$$

$$\text{with} \quad W(x) = -a_2 \coth x + \frac{b_2}{a_2}, \qquad a_2 > 0, \quad b_2 > 0 \tag{45}$$

For the sake of convenience, we put $g(x) = 1$. This simplifies (39) to

$$\begin{aligned} &\left(\lambda_1^2 c_1 - \lambda_1\right) f'(x) + 2\lambda_1\delta_1 f(x) + \lambda_1^2 c_2 + \delta_1^2 \\ &\quad = \left(\frac{(1-4\alpha\beta)\lambda_2^2 c_1}{(1-\alpha-\beta)^2} - \frac{\lambda_2}{1-\alpha-\beta)}\right) f'(x) + \frac{2\lambda_2\delta_2(1-4\alpha\beta)}{(1-\alpha-\beta)^2} f(x) \\ &\qquad + \frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}\left(\lambda_2^2 c_2 + \delta_2^2\right) \end{aligned} \tag{46}$$

Equating like terms on both sides, the unknown parameters λ_1, δ_1 are expressed in terms of the known ones λ_2, δ_2 through the following:

$$\lambda_1^2 c_1 - \lambda_1 = \frac{\lambda_2^2 c_1(1-4\alpha\beta)}{(1-\alpha-\beta)^2} - \frac{\lambda_2}{1-\alpha-\beta} \tag{47}$$

or, more explicitly,

$$\lambda_1 = \frac{1 \pm \sqrt{1+4\sigma_-}}{2c_1} \tag{48}$$

where

$$\sigma_- = \frac{\lambda_2^2 c_1(1-4\alpha\beta)}{(1-\alpha-\beta)^2} - \frac{\lambda_2}{1-\alpha-\beta} \tag{49}$$

and

$$\delta_1 = \frac{\lambda_2\delta_2}{\lambda_1}\frac{1-4\alpha\beta}{1-\alpha-\beta} \tag{50}$$

Since λ_1 and λ_2 should be of the same sign, only the positive sign is allowed in the expression for λ_1 in (48). The pseudo supersymmetric partners $H_\pm$, expressed as,

$$H_\pm \psi^{(\pm)}(x) = E^{(\pm)}\psi^{(\pm)}(x) \tag{51}$$

or, more explicitly,

$$H_\pm(x) = (1-\alpha-\beta)\left\{-\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\alpha-\beta}{1-\alpha-\beta}W(x)\right)^2 + V_\pm(x)\right\} \tag{52}$$

have identical energies except for the ground state, with $V_\pm(x)$ for this class of potentials reducing to

$$V_\pm(x) = \left(\lambda_1^2 c_1 \pm \lambda_1\right) f'(x) + 2\lambda_1\delta_1 f(x) + \delta_1^2 + c_2\lambda_1^2 \tag{53}$$

Table 1. The partner potentials $V_\pm(x)$ for the Rosen–Morse I and II, and the Eckart potentials. $\varepsilon_n^{(\pm)}$ stands for $\varepsilon_n^{(\pm)} = \frac{1}{E_n^{(\pm)}}$, $E_n^{(+)} = E_{n+1}^{(-)}$.

Model	$f(x)$	$W(x)$	$V_\pm(x)$	$\varepsilon_n^{(-)}$
Rosen–Morse I			$-\left(a_2^2 - \frac{b_2^2}{a_2^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$	$-\left(a_2^2 - \frac{b_2^2}{a_2^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$
$\lambda_2 = -a_2$	$\cot x$	$-a_2 \cot x$	$+a_1(a_1 \pm 1)\csc^2 x$	$+(a_1+n)^2$
$\delta_2 = -\frac{b_2}{a_2}$		$-\frac{b_2}{a_2}$	$+2b_2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}\cot x$	$-\frac{b_1^2}{(a_1+n)^2}$
$c_1 = c_2 = -1$				$n = 0, 1, 2, \ldots$
Rosen–Morse II			$\left(a_2^2 + \frac{b_2^2}{a_2^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$	$\left(a_2^2 + \frac{b_2^2}{a_2^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$
$\lambda_2 = a_2$	$\tanh x$	$a_2 \tanh x$	$-a_1(a_1 \pm 1)\mathrm{sech}^2 x$	$-(a_1-n)^2$
$\delta_2 = \frac{b_2}{a_2}$		$+\frac{b_2}{a_2}$	$+2b_2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}\tanh x$	$-\frac{b_1^2}{(a_1-n)^2}$
$c_1 = -1, c_2 = 1$				$n = 0, 1, 2, \ldots < a_1$
Eckart			$\left(a_2^2 + \frac{b_2^2}{a_2^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$	$\left(a_2^2 + \frac{b_2^2}{a_2^2}\right)\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$
$\lambda_2 = -a_2$	$\coth x$	$-a_2 \coth x$	$+a_1(a_1 \pm 1)\mathrm{csch}^2 x$	$-(a_1+n)^2$
$\delta_2 = \frac{b_2}{a_2}$		$+\frac{b_2}{a_2}$	$-2b_2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}\coth x$	$-\frac{b_1^2}{(a_1+n)^2}$
$c_1 = -1, c_2 = 1$				$n = 0, 1, 2, \ldots$

Table 2. The (unnormalized) solutions of the original Hamiltonian H_-, for the Rosen–Morse I and II, and the Eckart potentials. The solutions of their respective partners H_+ can be obtained by applying the transformation $\psi_n^{(+)}(x) = \rho^{-1}\phi_n^{(+)}(x)$, where $\phi_n^{(+)}$ are the solutions of the supersymmetric partner Hamiltonian h_+. In the expression for $\psi_n^{(-)}$, the different parameters stand for $\mu_1 = \alpha_1\mu = \alpha_1\frac{\alpha-\beta}{1-\alpha-\beta}$, $\mu_2 = \frac{b_1}{a_1}\mu = \frac{b_1(\alpha-\beta)}{a_1(1-\alpha-\beta)}$

Model	y	$s_\pm$	$\psi_n^{(-)}$
Rosen–Morse I	$i\cot x$	$-a_1 - n \pm \mathrm{i}\frac{b_1}{a_1+n}$	$\mathrm{e}^{(\frac{b_1}{a_1+n}-\mu_1)x}\sin^{a_1+n+\mu_2}x\, P_n^{(s_+,s_-)}(y)$
Rosen–Morse II	$\tanh x$	$a_1 - n \pm \frac{b_1}{a_1-n}$	$(1-y)^{\frac{s_+-\mu_1}{2}}(1+y)^{\frac{s_--\mu_1}{2}}\mathrm{e}^{\mu_2 x}P_n^{(s_+,s_-)}(y)$
Eckart	$\coth x$	$\pm\frac{b_1}{a_1+n} - n - a_1$	$(y-1)^{\frac{s_++\mu_1}{2}}(y+1)^{\frac{s_-+\mu_1}{2}}\mathrm{e}^{\mu_2 x}P_n^{(s_+,s_-)}(y)$

The potentials falling in this category are listed below. In each case the form of $w(x)$ is similar to that of $W(x)$, with a_2 and b_2 being replaced by a_1 and b_1. The unknown parameters a_1 and b_1 are obtained in terms of the known ones a_2 and b_2 from expressions (48), (49) and (50). It can be shown that the eigen energies of the positive and the negative sector are related through

$$E_n^{(+)} = E_{n+1}^{(-)}, \qquad \text{with} \qquad E^{(\pm)} = (1-\alpha-\beta)\varepsilon^{(\pm)}, \qquad n = 0, 1, 2, \ldots \tag{54}$$

The partner potentials $V_\pm(x)$ are given in table 1 while the (unnormalized) solutions of the original Hamiltonian H_- are given in table 2. The solutions of its partner H_+ can be obtained by applying the transformation

$$\psi_n^{(+)}(x) = \rho^{-1}\phi_n^{(+)}(x) \tag{55}$$

where $\phi_n^{(+)}$ are the solutions of the supersymmetric partner Hamiltonian h_+. In the expression for $\psi_n^{(-)}$, the different parameters stand for

$$\mu_1 = a_1\mu = a_1\frac{\alpha-\beta}{1-\alpha-\beta}, \qquad \mu_2 = \frac{b_1}{a_1}\mu = \frac{b_1(\alpha-\beta)}{a_1(1-\alpha-\beta)} \tag{56}$$

It is evident from the explicit expressions for $\psi_n^{(-)}(x)$ that for its well-defined behaviour, α and β must obey additional constraints; e.g., for the Rosen–Morse II and Eckart models,

$$\alpha < \beta \tag{57}$$

while the Rosen–Morse I model requires

$$a_1 + n + \mu_2 > 0, \qquad \frac{b_1}{a_1+n} < \mu_1 \tag{58}$$

which, in turn, implies

$$\alpha > \beta \tag{59}$$

4.2. *Case 2: $f^2(x) = c_1 + c_2 g^2(x)$, $f'(x) = c_3 g^2(x)$, $g'(x) = c_4 f(x)g(x)$ with c_1, c_2, c_3, c_4 constants*

The models falling in this category are based on the:

(1) *Scarf I (trigonometric) potential*

$$V(x) = k_1\tan^2 x - k_2\sec x\tan x, \qquad -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \tag{60}$$

$$\text{with} \quad W(x) = \lambda_2\tan x - \delta_2\sec x \tag{61}$$

(2) *Scarf II (hyperbolic) potential*

$$V(x) = k_1\,\text{sech}^2 x + k_2\,\text{sech}\, x\tanh x, \qquad -\infty \leqslant x \leqslant \infty \tag{62}$$

$$\text{with} \quad W(x) = \lambda_2\tanh x + \delta_2\,\text{sech}\, x \tag{63}$$

(3) *Pöschl–Teller potential*

$$V(x) = k_1\,\text{cosech}^2 x - k_2\coth x\,\text{cosech}\, x, \qquad 0 \leqslant x \leqslant \infty \tag{64}$$

$$\text{with} \quad W(x) = \lambda_2\tanh x - \delta_2\,\text{cosech}\, x, \qquad \lambda_2 < \delta_2 \tag{65}$$

Thus (39) gets simplified to

$$\lambda_1^2 c_1 + \left(\lambda_1^2 c_2 + \delta_1^2 - \lambda_1 c_3\right) g^2(x) + (2\lambda_1\delta_1 - \delta_1 c_4) f(x)g(x) + \lambda_1^2 c_1$$

$$= \left[\frac{(1-4\alpha\beta)}{(1-\alpha-\beta)^2}\left(\lambda_2^2 c_2 + \delta_2^2\right) - \frac{\lambda_2 c_3}{1-\alpha-\beta)}\right] g^2(x) + \frac{\lambda_2^2 c_1(1-4\alpha\beta)}{(1-\alpha-\beta)^2} \tag{66}$$

$$+ \left[2\lambda_2\delta_2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2} - \frac{\delta_2 c_4}{1-\alpha-\beta}\right] f(x)g(x) \tag{66}$$

Equating like terms on both sides, the unknown parameters λ_1, δ_1 are obtained by solving the following two coupled equations simultaneously:

$$\delta_1^2 + \lambda_1^2 c_2 - \lambda_1 c_3 = \left(\lambda_2^2 c_2 + \delta_2^2\right)\frac{(1-4\alpha\beta)}{(1-\alpha-\beta)^2} - \frac{\lambda_2 c_3}{1-\alpha-\beta} \tag{67}$$

$$2\lambda_1\delta_1 - \delta_1 c_4 = 2\lambda_2\delta_2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2} - \frac{\delta_2 c_4}{1-\alpha-\beta} \tag{68}$$

Once again, the pseudo supersymmetric partner Hamiltonians $H_\pm$, given by (52), have identical energies except for the ground state, with $V_\pm(x)$ for this class of potentials assuming

Table 3. The partner potentials $V_\pm(x)$ for the Scarf I and II, and the Pöschl–Teller potentials. $\varepsilon_n^{(\pm)}$ is the same as that defined in table 1.

Model	$f(x)$	$g(x)$	$V_\pm(x)$	$\varepsilon_n^{(-)}$
Scarf I			$-\lambda_2^2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2} - \lambda_1^2$	$-\lambda_2^2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$
$c_1=-1, c_2=1$	$\tan x$	$-\sec x$	$+\left(\delta_1^2+\lambda_1^2\pm\lambda_1\right)\sec^2 x$	$-\lambda_1^2+(\lambda_1+n)^2$
$c_3=1, c_4=1$			$-\delta_1\,(2\lambda_1\pm 1)\sec x\tan x$	$n=0,1,2,\ldots$
Scarf II			$\lambda_2^2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}+\lambda_1^2$	$\lambda_2^2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$
$c_1=1, c_2=-1$	$\tanh x$	$\mathrm{sech} x$	$+\left(\delta_1^2-\lambda_1^2\pm\lambda_1\right)\mathrm{sech}^2 x$	$+\lambda_1^2-(\lambda_1-n)^2$
$c_3=1, c_4=-1$			$+\delta_1(2\lambda_1\mp 1)\mathrm{sech} x\tanh x$	$n=0,1,2,\ldots<\lambda_1$
Pöschl–Teller			$\lambda_2^2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}+\lambda_1^2$	$\lambda_2^2\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}$
$c_1=1, c_2=1$	$\coth x$	$-\mathrm{csch} x$	$+(\delta_1^2+\lambda_1^2\pm\lambda_1)\mathrm{csch}^2 x$	$+\lambda_1^2-(\lambda_1-n)^2$
$c_3=-1, c_4=-1$			$-\delta_1(2\lambda_1\mp 1)\mathrm{csch} x\coth x$	$n=0,1,2,\ldots<\lambda_1$

Table 4. The (unnormalized) solutions of the original Hamiltonian H_-, for the Scarf I and II, and the Pöschl–Teller potentials, with $\mu_1=\lambda_2\mu,\ \mu_2=\delta_2\mu$. The solutions of their respective partners H_+ can be obtained in the same way as given in table 2.

Model	y	$s_\pm$	$\psi_n^{(-)}$
Scarf I	$\sin x$	$\lambda_1\pm\delta_1-\frac{1}{2}$	$(\sec x+\tan x)^{-\mu_2}(1-y)^{\frac{\lambda_1-\delta_1-\mu_1}{2}}(1+y)^{\frac{\lambda_1+\delta_1-\mu_1}{2}}P_n^{(s_-,s_+)}(y)$
Scarf II	$\sinh x$	$\pm i\delta_1-\lambda_1-\frac{1}{2}$	$(1+y^2)^{\frac{\mu_1-\lambda_1}{2}}\mathrm{e}^{(\mu_2-\delta_1)\tan^{-1}y}P_n^{(s_+,s_-)}(y)$
Pöschl–Teller	$\cosh x$	$\pm\delta_1-\lambda_1-\frac{1}{2}$	$(y-1)^{\frac{\delta_1-\lambda_1+\mu_1}{2}}(y+1)^{\frac{-\delta_1-\lambda_1+\mu_1}{2}}\mathrm{e}^{\mu_2 x}P_n^{(s_+,s_-)}(y)$

the form

$$V_\pm(x)=\lambda_1^2c_1+\left(\lambda_1^2c_2+\delta_1^2\pm\lambda_1c_3\right)g^2(x)+(2\lambda_1\delta_1\pm\delta_1c_4)\,f(x)g(x)+\lambda_2^2c_1\frac{1-4\alpha\beta}{(1-\alpha-\beta)^2} \tag{69}$$

In each of the cases of the three different models in this category, the form of $w(x)$ is similar to that of $W(x)$, with λ_2 and δ_2 being replaced by λ_1 and δ_1. The unknown parameters λ_1 and δ_1 are obtained in terms of the known ones λ_2 and δ_2 from expressions (48), (49) and (50). The pseudo supersymmetric partner Hamiltonians of the form given in (52), have energies $E^{(\pm)}$, related to $\varepsilon^{(\pm)}$ through (54). The partner potentials $V_\pm(x)$ are given in table 3 while the solutions are given in table 4, with

$$\mu_1=\lambda_2\mu,\qquad \mu_2=\delta_2\mu \tag{70}$$

From the explicit expressions for the solutions, it is evident that well defined behaviour is assured only when the parameters satisfy additional constraints. For example, for the Pöschl–Teller model, this condition reduces to

$$\mu_2<0\qquad \text{i.e.}\quad \alpha<\beta$$

4.3. Case 3: $g(x)=1$, and $f'(x)=kf(x)$, with $k=-1$

The *Morse potential*, given by

$$V(x)=a_1^2+b_1^2\exp(-2x)-b_1(2a_1+1)\exp(-x),\qquad -\infty\leqslant x\leqslant\infty \tag{71}$$

belongs to this class of potentials, with

$$W(x) = a_2 - b_2 \exp(-x) \tag{72}$$

Thus, for this particular model, $\lambda_2 = -b_2, \delta_2 = a_2$, with $f(x) = \exp(-x)$, so that equation (39) reduces to

$$b_1 = b_2 \frac{\sqrt{1-4\alpha\beta}}{1-\alpha-\beta} \tag{73}$$

$$a_1 = \frac{1}{2b_1} \left\{ \frac{b_2}{(1-\alpha-\beta)^2} \left[2a_2\,(1-4\alpha\beta) + (1+\alpha+\beta)\right] - b_1 \right\} \tag{74}$$

Thus

$$V_\pm(x) = a_1^2 + b_1^2 \exp(-2x) - b_1\,(2a_1 \mp 1)\exp(-x) + a_2^2 \frac{1-4\alpha\beta}{(1-\alpha-\beta)^2} \tag{75}$$

admit energies

$$\begin{aligned} \varepsilon_n^{(-)} &= a_1^2 - (a_1 - n)^2 + a_2^2 \frac{1-4\alpha\beta}{(1-\alpha-\beta)^2}, \qquad n = 0, 1, 2, \ldots < a_1 \\ \varepsilon_n^{(+)} &= \varepsilon_{n+1}^{(-)} \end{aligned} \tag{76}$$

The solutions of the original non Hermitian Hamiltonian H_- are given by

$$\psi_n^{(-)}(x) \approx y^{\lambda_1-\mu_1-n}\, \mathrm{e}^{(\frac{\mu_2}{\delta_1}-1)\frac{y}{2}} L_n^{2\lambda_1-2n}(y) \tag{77}$$

where μ_1 and μ_2 are defined in equation (70) and

$$y = 2\delta_1\, \mathrm{e}^{-x} \tag{78}$$

4.4. *Case 4:* $g(x) = 1$, and $f(x) = x$

These values represent the *Shifted Oscillator*, denoted by the potential

$$V(x) = \frac{a^2}{4}\left(x - \frac{2b}{a}\right)^2 - \frac{a}{2}, \qquad \infty \leqslant x \leqslant \infty \tag{79}$$

with

$$W(x) = \tfrac{1}{2} a_2 x - b_2 \tag{80}$$

Proceeding in a similar fashion, and assuming $w(x)$ to be of the same form as $W(x)$, with a_2, b_2 replaced by a_1, b_1, we obtain the following results:

$$a_1 = \frac{a_2\sqrt{1-4\alpha\beta}}{1-\alpha-\beta}, \qquad b_1 = \frac{b_2\sqrt{1-4\alpha\beta}}{1-\alpha-\beta} \tag{81}$$

so that

$$V_\pm(x) = \frac{1}{4}a_1^2\left(x - \frac{2b_1}{a_1}\right)^2 \pm \frac{a_1}{2} - \frac{a_2}{2\,(1-\alpha-\beta)} \tag{82}$$

with energy

$$\begin{aligned} \varepsilon_n^{(-)} &= a_1 n - \frac{a_2}{2\,(1-\alpha-\beta)}, \qquad n = 0, 1, 2, \ldots \\ \varepsilon_n^{(+)} &= \varepsilon_{n+1}^{(-)} \end{aligned} \tag{83}$$

Writing the solutions of H_- directly

$$\psi_n^{(-)}(x) \approx \mathrm{e}^{(\mu_1-a_1)\frac{x^2}{4}+(\mu_2-b_1)x} H_n(y) \tag{84}$$

where $H_n(y)$ are the Hermite polynomials, $y = \sqrt{\frac{a_1}{2}}\left(x - \frac{2b_1}{a_1}\right)$ and $\mu_1 = \mu a_2$, $\mu_2 = \mu b_2$. It can be checked that for the solutions to behave properly in the entire interval, the parameters should obey the condition $|\alpha + \beta| < 1$.

5. Conclusions

To conclude, we have developed a formalism to find an isospectral partner Hamiltonian H_+ of the generalized Swanson model, viz., $H_- = \mathcal{A}^\dagger\mathcal{A} + \alpha\mathcal{A}^2 + \beta\mathcal{A}^{\dagger 2}$. Though both the initial Hamiltonian H_- as well as its partner H_+ are non Hermitian, nevertheless they have real energies for certain range of parameter values. It is observed that $H_\pm$ form a pair of pseudo super symmetric partners of a pseudo super Hamiltonian $\mathcal{H}$, and share identical energies except for the ground state. Furthermore, the same similarity transformation operator ρ maps the pair of non Hermitian Hamiltonians $H_\pm$ to their respective Hermitian counterparts $h_\pm$, through $H_\pm = \rho^{-1}h_\pm\rho$, and these Hermitian maps form a pair of supersymmetric partners, generated by supercharges $q, q^\dagger$. The pseudo super charges Q, $Q^\sharp$ generating the pseudo super algebra of $\mathcal{H}$ are also related to $q, q^\dagger$ through the similarity transformation: $Q = \rho^{-1}q\rho$.

Since we have introduced non Hermiticity through an imaginary vector potential, the Hermitian maps $h_\pm$ obtained by similarity transformation are Schrödinger operators comprising of the standard kinetic term plus a local real Hermitian potential. It may be mentioned here that though two Hamiltonians may be related by similarity transformations, yet they can reveal different physical aspects of the dynamical system. In fact, for a particular class of potentials, certain physical properties are expected to emerge more distinctly in the non Hermitian framework. For example, *exceptional points*, or branch-point singularities of the spectrum and eigenfunctions, are associated with non Hermitian operators [17]. However, when one goes from the non Hermitian to the corresponding Hermitian picture, the exceptional points are lost, and consequently the entire information related to such phenomena. Additionally, though the super symmetric partners $h_\pm$ of a Hermitian Hamiltonian can always be mapped to non Hermitian ones (say $H_\pm$) by a similarity transformation, there is absolutely no way to determine whether $H_\pm$ are isospectral or not. This is due to the fact that to write the pseudo Hermitian partner Hamiltonian H_+ in terms of the generalized annihilation and creation operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ is still an open problem. As a result, while $h_\pm$ look similar in appearance (being expressed in terms of the creation and annihilation operators $A^\dagger$ and A), $H_\pm$ are not look-alikes. Nevertheless, we have been able to express $H_\pm$ in terms of the operators $D_\pm$, thus proving them to be related by pseudo super symmetry, sharing identical energies, barring the ground state.

We have applied our formalism successfully to all the known classes of shape-invariant models where the parameters of the original potential and its shape-invariant partner are related through translation. A general formula has been obtained for generating the respective pseudo supersymmetric partner Hamiltonians for such cases. Interestingly, the wave functions are automatically normalized following the normalization criterion for pseudo Hermitian systems [13]. We have intentionally left out the 3-dimensional shape-invariant models falling in this category, viz. 3-dimensional oscillator and Coulomb models, as we have restricted this work to deal with one-dimensional systems only. However, the radial part of these models can be studied in this framework, with $0 \leqslant r \leqslant \infty$.

This work deals with real Hamiltonians that are nevertheless non Hermitian. We can make a straightforward extension of our formalism to map a complex non Hermitian Hamiltonian H to a Schrödinger Hamiltonian which is also complex but $\mathcal{PT}$ symmetric. However, in such a case H will be weakly pseudo Hermitian [18]. Finally we would like to note that in this work we have studied shape-invariant models with unbroken supersymmetry. It would be interesting to study models with broken supersymmetry, too, in this framework.

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